The Critical Amplitude for Bifurcation Points of the Logistic Map

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The phenomenon called critical slowing down is studied for some analytically tractable bifurcation points of the logistic map. The critical amplitude and the critical amplitude ratio are introduced to describe the critical behaviors more precisely when the critical exponents are identically unity for two critical behaviors which are to be compared. Since Hao predicted unity of the critical exponent for 1-d maps assuming exponential damping of the distance from the attractor, the assumption is checked. The result is that the shrinkage of the valid region of the assumption occurs as the adjustable parameter approaches one of the bifurcation points.

Key words: Bifurcation, Logistic map, Critical slowing down, Critical exponent, Critical amplitude.

1. Introduction

When a dynamical system shows bifurcation phenomena, it is known that one observes so-called critical slowing down; an orbit takes more and more time or iterations to settle down at an attractor as the adjustable parameter value gets closer and closer to that of the bifurcation point [1]. Hao showed that the critical exponent β (Δ in his notation) for this phenomenon is unity for all the period-doubling bifurcation points of a one-humped one-dimensional map, *assuming* that the distance between the orbit and an attractor diminishes exponentially [2]. For some unknown reasons he restricted his treatment to the lower sides of the bifurcation points, saying that the phenomenon is one-sided, i.e., the critical slowing down is seen when the adjustable parameter closes in on the bifurcation point from the less-bifurcated side. However, for, e.g., the logistic map, which is the most widely studied generic example of the 1-d maps and defined as

$$x_{n+1} = a\,x_n(1-x_n), \quad (0 \le x \le 1), \quad (0 \le a \le 4), \qquad (1)$$

it can easily be confirmed numerically that the critical slowing down occurs also for the upper sides of the bifurcation points, i.e., the parameter approaches the bifurcation point from the more-bifurcated side. This occurs not only for the period-doubling bifurcation points but also for both sides of the transcritical bifurcation point at $a = 1$. In Fig. 1 a is plotted the number of iterations to settle into the corresponding attractors against the adjustable parameter a for $a = 1$ and in Fig. 1 b for the first period-doubling bifurcation point $a = 3$. Since the attractors for both sides of these bifurcation points can be expressed analytically for these cases, no other parameters than the error tolerance ε are necessary when the numerical calculations are carried out.

Hao's theory relies on the assumption of exponential diminishing of the distance between the orbit and the attractor. This assumption makes the problem equivalent to treat the Lyapunov exponent λ. We show that by using analytic expressions for λ in (1), the number of steps to the attractor in numerical calculations can be expressed exactly, and by observing the deviation of the numerical results from the exact expression one can estimate the valid region of the exponential diminishing assumption. It is seen that, as the parameter value gets closer to that of the bifurcation point, the valid region becomes narrower. For $a \to 3_-$ a reliable value of the exponent for this shrinking phenomenon can be obtained numerically.

Also, by utilizing the analytic expressions of the Lyapunov exponent λ for the parameter range $0 < a < 1 + \sqrt{6}$,we readily show that the critical exponents are unity for the critical behaviors at $a = 1_\pm$, $3_\pm$ and $(1+\sqrt{6})_-$. Hence one sees that the asymmetric behaviors with respect to the bifurcation point $a = 3$ manifested in Fig. 1 b are not due to the difference in the exponent but to the difference in the critical amplitude which frequently appeared in statistical-mechanical studies of phase transitions back in the 1970's [3]. If the critical exponents are identically unity, the critical amplitude is given by the inverse of

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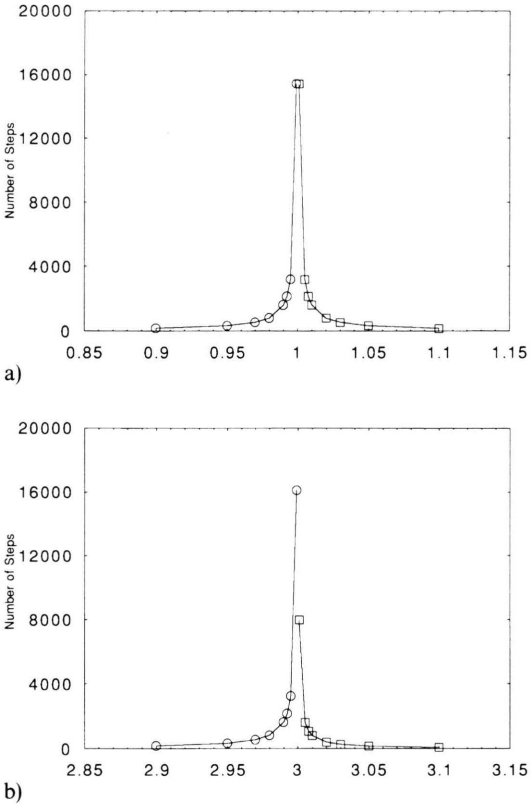

Fig. 1. a) The number of steps for an orbit to settle into the ε-neighborhood of the attractor below and above the transcritical bifurcation point $a = 1$ of the logistic map (1). The initial distance ε_0 is set at 0.001 for all the runs, and $\varepsilon = 10^{-10}$. b) The number of steps for the first period-doubling bifurcation point $a = 3$. The values of ε and ε_0 are the same as those used in a).

the derivative of the Lyapunov exponent λ with respect to the adjustable parameter a.

In the next section we check the validity of the exponential damping assumption, taking $a = 3_-$ for (1) as an example, and recognize that the valid region shrinks as $(3 - a) \to 0$. In Sect. 3 the critical exponents and the amplitudes that can be derived exactly for (1) are given. A summary and discussion are given in Sect. 4.

2. Exact Number of Steps

If we assume, as Hao did, that for an orbit starting from the initial point close to the attractor the distance from the attractor diminishes exponentially, the distance ε_n for the n-th step is written as

$$\varepsilon_n = \varepsilon_0 \, e^{-n/\tau} , \tag{2}$$

where ε_0 is the initial distance from the attractor and τ is the decay constant. Equation (2) can also be regarded as the definition of the Lyapunov exponent λ which is conventionally expressed as

$$\delta_n = \delta_0 \, e^{\lambda n} , \tag{3}$$

where δ represents the distance between the two orbits under consideration. In (2), therefore, the distance between an orbit starting from the attractor itself and the other orbit starting from a nearby point is treated. The relation between our τ and the Lyapunov exponent is, of course,

$$\tau = -\frac{1}{\lambda} . \tag{4}$$

This relation can be used to verify the validity of the assumption of the exponential damping stated above, if the exact expression for the Lyapunov exponent is available.

When performing numerical calculations of the number of steps s, we specify the initial distance ε_0 between the attractor and the initial point and count the number of steps until the orbit enters the ε-neighborhood of the exact value of the attractor for the first time. This count of the steps is the result s for the tolerance ε which is the ultimate resolution of the calculation. Nothing to do with the assumption of the exponential damping comes in.

On the other hand, employing the exponential damping assumption means that the distance diminishes according to (2), hence we count the number of steps s that satisfy the inequality

$$\varepsilon \geq \varepsilon_0 \, e^{-s/\tau} , \tag{5}$$

or more explicitly

$$s \geq -\tau \ln(\varepsilon/\varepsilon_0) \tag{6}$$

for the first time.

For the logistic map (1), the Lyapunov exponent for the parameter range $0 < a < 1 + \sqrt{6}$ can be written exactly as

$$\lambda = \ln|a| , \quad (0 < a < 1) , \tag{7}$$

$$\lambda = \ln|2 - a| , \quad (1 < a < 3) \tag{8}$$

and

$$\lambda = \tfrac{1}{2} \ln|-a^2 + 2a + 4| , \quad (3 < a < 1 + \sqrt{6}). \tag{9}$$

Substituting these expressions into (6) via (4), we have

$$s \geq \frac{\ln(\varepsilon/\varepsilon_0)}{\ln|a|} \quad (0<a<1)\,, \qquad (10)$$

$$s \geq \frac{\ln(\varepsilon/\varepsilon_0)}{\ln|2-a|} \quad (1<a<3) \qquad (11)$$

and

$$s \geq \frac{\ln(\varepsilon/\varepsilon_0)}{\ln|-a^2+2a+4|}\,, \quad (3<a<1+\sqrt{6})\,. \quad (12)$$

Therefore, under the assumption of exponential damping the number of steps s should be given by the smallest integer that satisfies the inequality (10), (11) or (12), depending on the parameter range. In Table 1 we compare the results with those of numerical calculation for $a = 2.99$ and $\varepsilon = 10^{-10}$ as an example. The exponential damping assumption is precisely satisfied as far as the initial distance from the attractor is less than 10^{-2}. Similar results are obtained for the vicinity of other exactly tractable bifurcation points, namely, $a = 1_{\pm}, 3_{+}$ and $(1+\sqrt{6})_{-}$.

However, if we further decrease $|a - a_k|$, where a_k refers to the bifurcation point under consideration, we see that the exponential damping region shrinks as shown in Fig. 2 for $a = 3_{-}$. Plotting each curve shown in Fig. 2 separately as a full-scale graph (not shown here), one sees that the validity region of the exponential damping is reduced by roughly an order when $|a - a_k|$ is reduced by 2 orders. This can be confirmed and even the exponent for this narrowing phenomenon can be determined for $a = 3_{-}$. Since we are dealing with the departure from the exponential formula, we must first specify how to measure the deviation. There may be various ways to measure it, but

Table 1. Comparison of numerically calculated numbers of steps s with those calculated by (11) for varying initial distances ε_0 for $a = 2.99$.

ε_0	Numerical s	(11)
10^{-1}	1944	2062
10^{-2}	1828	1833
10^{-3}	1604	1604
10^{-4}	1375	1375
10^{-5}	1146	1146
10^{-6}	917	917
10^{-7}	688	688
10^{-8}	459	459
10^{-9}	230	230
10^{-10}	1	1

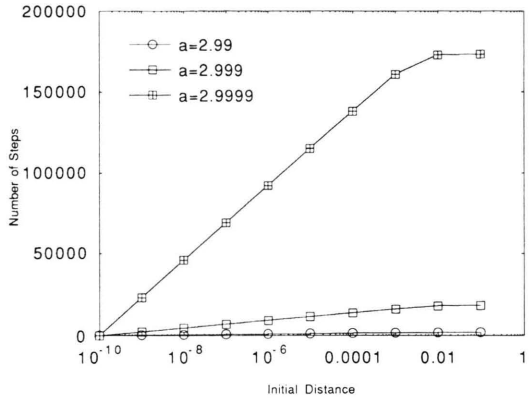

Fig. 2. The number of steps for an orbit to settle into the ε-neighborhood ($\varepsilon = 10^{-10}$) for varying initial distances ε_0 for $a = 3_{-}$. Deviations from the straight lines indicate that the exponential damping assumption does no longer hold there.

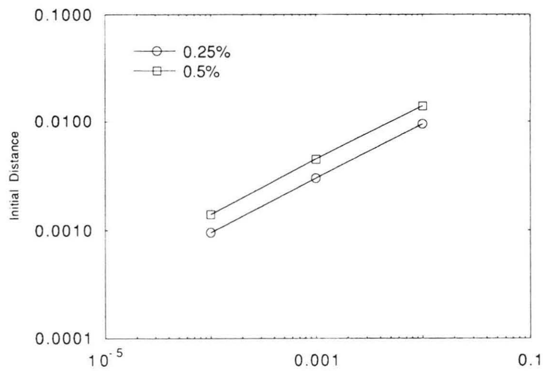

Fig. 3. The initial distance $\varepsilon_0\,(3-a)$ at which the indicated deviations (see text) from the number of steps predicted by the exponential damping assumption are detected for $a = 3_{-}$.

here we adopt the following as our criterion. Let us denote the numerical s in Table 1 as S and the value obtained from (11) as s_{H}. When the ratio $(S - s_{\mathrm{H}})/s_{\mathrm{H}}$ exceeds some specified value, here we take 0.25% and 0.5%, say that at that distance ε_0, S departs from s_{H}. Thus we measure the deviation as we vary $(3-a)$. The results are shown in Fig. 3 and show that the exponent for this phenomenon is 1/2. This result is independent of the tolerance ε; we confirmed the region $10^{-8} > \varepsilon > 10^{-12}$. For $a \to 1_{\pm}$ straight lines are also obtained in the $\log|a-1| - \log\varepsilon_0$ plane, and the slope for the line corresponding to 0.25% is about 0.95 and that for 0.5% is about 1.08. We thus conjecture that the exponents for the shrinking of the exponential damping are 1 for $a \to 1_{-}$ and $a \to 1_{+}$, with less confi-

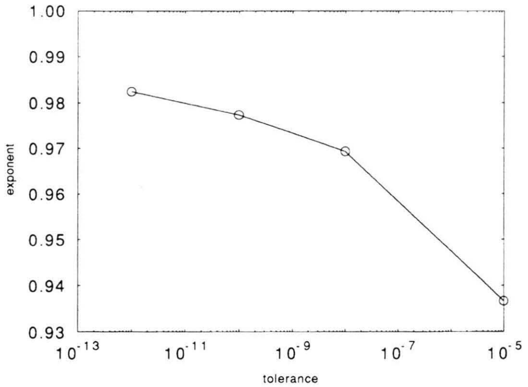

Fig. 4. The critical exponent derived from the slope of the $\log\langle s\rangle - \log(3-a)$ plot for varying tolerance ε. The range of a used is $0.0001 < (3-a) < 0.01$.

dence, however, than in the case of $a \to 3_-$ above. For $a \to 3_+$ and $a \to (1+\sqrt{6})_-$, however, straight lines are not obtained but slightly concave curves, which make us unable to guess the exponent. But the fact that the valid regions of the exponential damping assumption decrease is never altered also for these cases.

Ordinarily the quantities of interest such as the Lyapunov exponent and the critical exponent are calculated as the averaged quantities over all the possible initial conditions. If we calculate the average number of steps $\langle s\rangle$ over 10^4 evenly distributed initial points for the case of Table 1, we have $\langle s\rangle = 1918.42$. This means that an overwhelming number of the initial points are *virtually* located at the distance $10^{-2} \sim 10^{-1}$ from the attractor or an overwhelming number of orbits enter the exponential damping region in a few steps, namely much faster than exponentail. If we derive the critical exponent for $a = 3_-$ from the slope of the graph of $\langle s\rangle$ against $|a-3|$ in the range $0.0001 < |a-3| < 0.01$ without discarding any initial steps, we obtain the results for varying ε shown in Figure 4. Due to that overwhelming number of orbits, the exponent becomes smaller than 1. If we use only the orbits within the exponential damping region, the exponent becomes almost unity even for the same range of $|a = 3|$ and for $10^{-12} < \varepsilon < 10^{-8}$.

3. Exponents and Amplitudes

Using the exact expressions for the Lyapunov exponent (7)–(9) and the relation (4), we can derive the expressions for τ in the respective critical regions by expanding the logarithms around the respective bifurcation points as

$$\tau \simeq \frac{1}{1-a} \quad (a \to 1_-)\,, \tag{13}$$

$$\tau \simeq \frac{1}{a-1} \quad (a \to 1_+)\,, \tag{14}$$

$$\tau \simeq \frac{1}{3-a} \quad (a \to 3_-)\,, \tag{15}$$

$$\tau \simeq \frac{1}{2(a-3)} \quad (a \to 3_+)\,, \tag{16}$$

$$\tau \simeq \frac{1}{\sqrt{6}(1+\sqrt{6}-a)} \quad (a \to (1+\sqrt{6})_-)\,. \tag{17}$$

The above results are valid as long as the exponential damping assumption holds. All the critical exponents are unity not only for the lower sides of the period-doubling bifurcation points ($a \to 3_-$ and $a \to (1+\sqrt{6})_-$) that are covered by Hao's theory, but also for both sides of the transcritical bifurcation point ($a \to 1_\pm$) and the upper side of the first period-doubling bifurcation point ($a \to 3_+$) that are not explicitly covered by his theory. We now see that, despite of the asymmetry seen in Fig. 1b for $a_1 = 3$, the critical exponents are identical, namely unity, for both sides of the bifurcation point. Then how can we distinguish the asymmetry quantitatively?

If τ can be expressed in the neighborhood of a bifurcation point a_k as

$$\tau \simeq \frac{A}{|a-a_k|^\beta}\,, \tag{18}$$

then the coefficient A is called the critical amplitude. The critical amplitude was introduced circa 1970 in the course of statistical-mechanical studies of phase transitions in order to describe critical behaviors more precisely in addition to the critical exponents. However, probably due to the lack of universality, it has not been emphasized so much as the critical exponents in the later studies of phase transitions. The present author considers, however, that when all the critical exponents are identical, such as in the case we are here dealing with, the critical amplitudes and their ratios are useful to describe the critical behaviors quantitatively. Especially when the exponents are unity, as in our situation for the logistic map, the amplitude A is given by

$$A = -\frac{1}{\mathrm{d}\lambda/\mathrm{d}a} \tag{19}$$

as long as the relation (4) holds.

We see from (13)–(16) that the symmetry in Fig. 1a is described by the identical amplitudes for both sides of the bifurcation point $a = 1$, and the asymmetric character observed in Fig. 1b is reflected by the different values of the amplitude A, i.e., 1 for the lower and 1/2 for the upper sides of the first period-doubling bifurcation point $a_1 = 3$.

In our exact treatment of the logistic equation the value of the critical amplitude itself is meaningful in that, e.g., the exact number of steps can be described by using it. But in practical cases the more important quantity might be the critical amplitude ratio R defined as

$$R = \frac{A_1}{A_2}, \tag{20}$$

where A_1 is the critical amplitude for a critical phenomenon 1 and A_2 is that for another critical phenomenon 2. (The critical exponents for 1 and 2 should be identical.) In our example the ratio R between the upper and the lower behaviors of the bifurcation points $a = 1$ and $a = 3$ are, respectively, 1 and 1/2, which are alternative descriptions of the difference between the phenomena shown in Fig. 1a and b.

4. Summary and Discussion

By fully utilizing the analytic expressions of the attractors and the Lyapunov exponents, we described quantitatively the critical amplitudes and their ratios for some bifurcation points of the logistic map (1) after confirming that the critical exponents for the critical behaviors under consideration are all unity. In order to legitimize the above derivation, we checked the valid region of the exponential damping assumption for the distance between the orbit and the attractor and found in fact that the valid region shrinks as the adjustable parameter closes in on the bifurcation point.

As we stated at the end of Sect. 2, the overwhelming number of orbits starting from possible initial points come into the exponential damping region very quickly, which means that the critical slowing down is apparently diminished when we consider the averaged number of steps over the possible initial points. By discarding the first few transient steps, we can eliminate this even for the case of the averaged steps. This settles the problem for situations where analytic expressions for the attractors are available like the ones we treated here. But for most of the situations we must first locate an attractor and then count the number of steps for an orbit to settle down at the ε-neighborhood of the attractor. To locate that attractor we employ, e.g., the Newton-Raphson method which includes another tolerance parameter. As the bifurcation point is approached, convergence of the method also comes to take a long time. Therefore, subtle results which are quite hard to distinguish from artifacts can appear depending on the two tolerance parameters. This will be discussed elsewhere.

[1] Although numerous textbooks and monographs on chaos and bifurcation are now available, not many of them explicitly discuss critical slowing down. A rare exception might be S. H. Strogatz, Nonlinear Dynamics and Chaos, Addison-Wesley, Reading MA, 1994.
[2] B. Hao, Phys. Lett. **86 A**, 267 (1981).
[3] See, e.g., review articles in C. Domb and M. S. Green (Eds.), Phase Transitions and Critical Phenomena, Vol. **3**, Academic Press, London 1974.